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Propagators for non-linear systems

D G Cacuci† and V Protopopescu

Engineering Physics and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA

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Abstract. A canonical formalism based on forward and backward propagators is developed for problems described by systems of general non-linear equations. These propagators are shown to yield the problem's solution by propagating exactly the bulk/surface/initial sources. They naturally generalise to non-linear problems the Green functions of linear theory. Unlike the customary Green functions, though, the forward and backward propagators depend parametrically and non-linearly on the problem's solution; however, the propagators themselves satisfy *linear* equations that can, in principle, be solved by methods of linear theory. Three examples, comprising both scalar and vector problems, are presented to highlight the main points underlying the application of this formalism.

1. Introduction

The established analytical methods for seeking solutions to non-linear equations, such as inverse scattering transform, Lax-pair representation and group theory, can only be used for special forms of operators and domains in phase spaces. On the other hand, geometrical and topological methods are being used in conjunction with detailed numerical calculations to study phenomena important and specific to non-linear problems such as the occurrence of bifurcations, shocks and chaos. The monographs by Calogero and Degasperis (1982) and by Chow and Hale (1982) cover a very large number of aspects and references in these two areas.

Recently, Cacuci *et al* (1988) have proposed a new formalism for solving general non-linear equations. This formalism is the natural generalisation to non-linear problems of the Green function formalism in linear theory, and is canonically and exactly applicable to any non-linear operator equation in phase-space domains where these respective Gâteaux derivatives exist. Fundamental to this formalism is the construction of advanced and retarded propagators; these propagators generalise the customary Green functions, to which they reduce, exactly, for linear problems.

We have presented the fundamental ideas underlying this formalism for scalar non-linear equations (Cacuci *et al* 1988). Cacuci and Karakashian (1988) applied this formalism to generalisations of Burgers and Korteweg-de Vries equations, noting its efficiency, accuracy and straightforward computational implementation.

The aim of this paper is to generalise the formalism to include the treatment of multicomponent (matrix) non-linear operator equations; in addition we present several

† Present address: Department of Chemical and Nuclear Engineering, 3323 Engineering II, University of California at Santa Barbara, Santa Barbara, CA 93106, USA.

important aspects of the basic theory that were not addressed in the original formulation. This material is presented in § 2, which introduces the propagators for the multicomponent non-linear systems, gives the system's solution in terms of these propagators, and demonstrates the uniqueness of this solution. Section 3 presents the closed-form integral equations satisfied by the propagators, while § 4 summarises the conclusions of our work. Specific illustrations of applying our formalism to scalar equations, a two-component (Carleman) system from the kinetic theory of gases and a matrix Riccati equation are presented in appendices A1, A2, and A3, respectively.

2. Propagators for non-linear systems

A general non-linear system can be represented in abstract form as

$$N(u)+\delta\Gamma(u)=f+\delta g \tag{2.1}$$

where $N(u)$ represents the non-linear equation, $\Gamma(u)$ represents the non-linear initial/boundary conditions, f represents the volume source, and g represents the boundary source (including initial conditions). The source term f includes the inhomogeneities of the non-linear operators, so we can consider without loss of generality that $N(0)=0$, and $\Gamma(0)=0$. The δ distributions multiplying the boundary terms in (2.1) allow a formally unified abstract treatment of both boundary conditions and operators. These δ distributions are associated with the direct boundary space of the problem (Coddington and Levinson 1955), being uniquely specified for each specific problem under consideration.

The system (2.1) represents an equation for an n-component vector $u=(u_1, u_2, \ldots, u_n)$; for simplicity, we consider each component u_i to be an element in the Hilbert space $L_2(\Omega)$ endowed with an inner product denoted by $\langle\ ,\ \rangle$; throughout this work, Ω denotes the set (including the time domain for time-dependent problems) that defines the phase space for (2.1). Since we include the boundaries in the formal treatment, Ω is a closed set, containing the boundaries of the phase space underlying the problem.

As before (Cacuci *et al* 1988), we require that the first Gâteaux derivatives (see e.g. Nashed 1970) of the operators appearing on the left-hand side of (2.1), defined as

$$[N'(u)+\delta\Gamma'(u)]h\equiv\{(\mathrm{d}/\mathrm{d}\varepsilon)[N(u+\varepsilon h)+\delta\Gamma(u+\varepsilon h)]\}_{\varepsilon=0} \tag{2.2}$$

exist; in our case, $N'(u)+\delta\Gamma'(u)$ is a $n\times n$ matrix whose (ij) element is the operator $\partial(N_i(u)+\delta\Gamma_i(u))/\partial u_j$. Each of these operators depend non-linearly on u but act linearly on the n-component vector h.

For an arbitrary n-component vector v, the operator adjoint to $N'(u)+\delta\Gamma'(u)$ is defined via the usual linear duality:

$$\langle[N'(u)+\delta\Gamma'(u)]h, v\rangle=\langle h, [N'^*(u)+\delta^*\Gamma'^*(u)]v\rangle. \tag{2.3}$$

In (2.3), $N'^*(u)$ is the *formal* adjoint of $N'(u)$, and $\Gamma'^*(u)$ includes all surface operators on v. Note in (2.3), that the operator δ^* is not the same δ distribution as in (2.1) or (2.2), but is a distribution associated with the adjoint boundary space (Coddington and Levinson 1955). To highlight this distinction, we use the symbolical notation δ^*.

(See appendices A1–A3 for examples.) The pair $\{N'^*(u), \Gamma'^*(u)\}$ is the adjoint of the pair $\{N'(u), \Gamma'(u)\}$; the explicit expression of the (ij)th element of the matrix $[N'^*(u)+\delta^*\Gamma'^*(u)]$ is $[\partial(N_j(u)+\delta\Gamma_j(u))/\partial u_i]^*$, obtained by taking the adjoint of each element of the transpose of $[N'(u)+\delta\Gamma'(u)]$. This follows naturally from the definition of the adjoint operator in $L_2(\Omega)\times\ldots\times L_2(\Omega)=(L_2(\Omega))^n$.

Following Cacuci *et al* (1987, 1988), we define the operators

$$L(u)h \equiv \int_0^1 N'(\varepsilon u)h\,\mathrm{d}\varepsilon \qquad \gamma(u)h \equiv \int_0^1 \Gamma'(\varepsilon u)h\,\mathrm{d}\varepsilon \tag{2.4}$$

and

$$L^*(u)v \equiv \int_0^1 [N'(\varepsilon u)]^* v\,\mathrm{d}\varepsilon \qquad \gamma^*(u)v \equiv \int_0^1 [\Gamma'(\varepsilon u)]^* v\,\mathrm{d}\varepsilon. \tag{2.5}$$

Note that the operators L, γ, L^* and γ^*, still act linearly on h and v, respectively, while retaining a non-linear parametric dependence on u. Note also the important relationship satisfied by $L(u)$ and $\gamma(u)$:

$$[L(u)+\delta\gamma(u)]u = N(u)+\delta\Gamma(u). \tag{2.6}$$

Relationship (2.6) underscores the important role played by the integrated operators L, L^*, γ, γ^*: in contradistinction to the variational operators N', N'^*, γ', γ'^*, it is the pair of integrated operators $\{L(u), \gamma(u)\}$ that restores exactly the original non-linear system (2.1) when applied to u. Furthermore, it follows from (2.3), (2.4) and (2.5) that the (ij)th component of the (matrix) operator $L^*(u)+\delta^*\gamma^*(u)$ is obtained by taking the adjoint of the (ji)th component of the (matrix) operator $L(u)+\delta\gamma(u)$:

$$(L^*(u)+\delta^*\gamma^*(\tilde{u}))_{ij} = [(L(u)+\delta\gamma(u))_{ji}]^*. \tag{2.7}$$

The backward (retarded) and forward (advanced) propagators are defined as the inverses of the operators $L(u)+\delta\gamma(u)$ and $L^*(u)+\delta^*\gamma^*(u)$, respectively:

$$[L(u)+\delta\gamma(u)]G_u = 1 \tag{2.8}$$

and

$$[L^*(u)+\delta^*\gamma^*(u)]G_u^* = 1 \tag{2.9}$$

where 1 denotes the unit operator. The propagators G_u and G_u^* are $n\times n$ matrices whose components are operators in $L_2(\Omega)$. Equations (2.8) and (2.9) can be written in terms of formal integral kernels as

$$[L(u(t))+\delta\gamma(u(t))]G(u(t);\,t,t') = \delta(t-t') \tag{2.10}$$

and

$$[L^*(u(t))+\delta^*\gamma^*(u(t))]G^*(u(t);\,t,t'') = \delta(t-t'') \tag{2.11}$$

where t is a shorthand notation for the generic variable in the domain Ω (including its boundaries).

Since the operators $L(u)+\delta\gamma(u)$ and $L^*(u)+\delta^*\gamma^*(u)$ act linearly on the respective propagators, the relationships between the propagators and the expression for the solution u in terms of these propagators can be derived, as previously noted (Cacuci *et al* 1988), in the same spirit as for the usual Green functions in linear theory (Butkovskiy 1982, Roach 1982). Thus, forming the inner products of (2.10) and (2.11) with $G^*(u(t); t, t'')$ and $G(u(t); t, t')$, respectively, and taking into account (2.7) leads to the reciprocity relation

$$G^*(u(t); t, t') = G(u(t'); t', t). \tag{2.12}$$

In component form, (2.12) implies that $(G_u)^*_{ij} = (G_u)_{ji}$.

The solution u of the original non-linear system (2.1) is obtained in terms of the forward propagator G^*_u as follows: firstly, by (2.1),

$$u = \langle u, \delta\rangle - \langle N(u) + \delta\Gamma(u), G^*_u\rangle + \langle f + \delta g, G^*_u\rangle$$

and secondly, by (2.11) and (2.6),

$$\begin{aligned} u &= \langle u, [L^*(u) + \delta^*\gamma^*(u)]G^*_u\rangle - \langle [L(u) + \delta\gamma(u)]u, G^*_u\rangle + \langle f + \delta g, G^*_u\rangle \\ &= \langle f + \delta g, G^*_u\rangle \end{aligned} \tag{2.13}$$

Using the reciprocity relationship (2.12) in (2.13) yields the solution u in terms of the backward propagator G_u as

$$u = \langle G_u, f + \delta g\rangle = \int_\Omega G(u(t); t, t')[f(t') + \delta g(t')]. \tag{2.14}$$

In the remainder of this section, we investigate the uniqueness of the representations (2.13) and (2.14) of the solution u in terms of the propagators G^*_u and G_u, respectively. Thus, consider that the system (2.1) admits a unique solution $\tilde{u}$, i.e. consider $\tilde{u}$ to be a solution branch free of bifurcations. We note that, by adding a homogeneous function of $u - \tilde{u}$ to (2.1), we obtain a different system which, however, still admits $\tilde{u}$ as a solution. Nevertheless, this addition modifies the form of the operators N, Γ and the sources f and g. Consequently, the operators L and γ, and the propagators G and G^* will have expressions that would differ from those obtained from the original system (2.1). The new set of propagators and sources thus obtained, though, should yield the same solution $\tilde{u}$. In the following, we give a formal proof that this is indeed the case.

We start with two equivalent forms of the system (2.1), namely

$$N_1(u) + \delta\Gamma_1(u) = \delta g_1 + f_1 \tag{2.15}$$

$$N_2(u) + \delta\Gamma_2(u) = \delta g_2 + f_2. \tag{2.16}$$

The system (2.15) is equivalent to (2.16) in the sense that it is obtained from (2.16) by a transformation that vanishes when applied to the (unique) solution $\tilde{u}$; thus, $\tilde{u}$ solves both (2.15) and (2.16). In addition, we require this transformation not to introduce new solutions (i.e. in addition to $\tilde{u}$); this requirement is consistent with our initial requirement that $\tilde{u}$ be a solution branch free of bifurcations, where the Gâteaux derivatives of N_1, N_2, Γ_1 and Γ_2 exist. Then the counterparts of (2.6)-(2.9) exist for both (2.15) and (2.16), and can be written as

$$L_1(u)u + \delta\gamma_1(u)u = \delta f_1 + g_1 \tag{2.17}$$

and

$$L_2(u)u + \delta\gamma_2(u)u = \delta f_2 + g_2 \tag{2.18}$$

while the respective propagators are obtained as the solutions of the systems

$$[L_1(\tilde{u})+\delta\gamma_1(\tilde{u})]G_1=1 \tag{2.19}$$

and

$$[L_2(\tilde{u})+\delta\gamma_2(\tilde{u})]G_2=1. \tag{2.20}$$

We need to show that the same solution $\tilde{u}$ is obtained by using either the propagator G_1 from (2.19) or the propagator G_2 from (2.20), i.e.

$$\tilde{u}=\langle G_i, f_i+\delta g_i\rangle=\langle f_i+\delta g_i, G_i^*\rangle \qquad i=1,2 \tag{2.21}$$

where the inner products have the same meaning as in (2.13) and (2.14). The notation for the proof of (2.21) can be simplified considerably by interpreting the propagators G_1 and G_2 simply as operators so that the first equality in (2.21) can be rewritten as

$$\tilde{u}=G_1(f_1+\delta g_1)=G_2(f_2+\delta g_2). \tag{2.22}$$

To prove (2.21), we start by considering (2.20) to be a linear perturbation of (2.19); the Dyson formula, which relates the resolvents of perturbed and unperturbed operators, can then be applied to (2.19) and (2.20) to obtain

$$G_2=G_1-G_1[L_2(\tilde{u})+\delta\gamma_2(\tilde{u})-L_1(\tilde{u})-\delta\gamma_1(\tilde{u})]G_2. \tag{2.23}$$

Suppose now that we express the solution in terms of the propagator G_2:

$$\tilde{u}=G_2(f_2+\delta g_2). \tag{2.24}$$

Taking into account (2.23), (2.24), (2.17) and (2.18), and using the linearity in G_i of the operators L_i $(i=1,2)$ allows us to perform the following sequence of operations:

$$\begin{aligned}\tilde{u}&=G_2(f_2+\delta g_2)\\&=G_1(f_2+\delta g_2)-G_1[L_2(\tilde{u})+\delta\gamma_2(\tilde{u})-L_1(\tilde{u})-\delta\gamma_1(\tilde{u})]G_2(f_2+\delta g_2)\\&=G_1(f_2+\delta g_2)-G_1[L_2(\tilde{u})+\delta\gamma_2(\tilde{u})-L_1(\tilde{u})-\delta\gamma_1(\tilde{u})]\tilde{u}\\&=G_1(f_2+\delta g_2)-G_1(f_2+\delta g_2-f_1-\delta g_1)\\&=G_1(f_1+\delta g_1)\\&=\tilde{u} \qquad \text{QED.}\end{aligned} \tag{2.25}$$

Remark. By construction, the operators $L(u)$, $L^*(u)$, $\gamma(u)$, and $\gamma^*(u)$ are canonically determined from $N(u)$ and $\Gamma(u)$. However, one could find operators distinct from the pair $\{L(u), \gamma(u)\}$ that would still give the right-hand side of (2.6) if they were replaced on the left-hand side of (2.6). All of these pairs of operators, $\{L_j(u), \gamma_j(u)\}$, and their respective adjoints, $\{L_j^*(u), \gamma_j^*(u)\}$ are equivalent to the pairs $\{L(u), \gamma(u)\}$ and $\{L^*(u), \gamma^*(u)\}$, respectively, in that they all give the correct solution u (to the original non-linear equation) in the form (2.14) provided, of course, that the respective boundary terms are properly taken into account. To prove this fact, one uses (2.23)–(2.25) for $f_2+\delta g_2=f_1+\delta g_1=f+\delta g$.

3. Integral equations for propagators

Consider (2.10) for a known vector u^0 and (2.11) for the actual solution $\tilde{u}$ of the original system (2.1), i.e.

$$[L(u^0)+\delta\gamma(u^0)]G_{u^0}=\delta(t-t') \tag{3.1}$$

and

$$[L^*(\tilde{u})+\delta^*\gamma^*(\tilde{u})]G^*_{\tilde{u}}=\delta(t-t''). \tag{3.2}$$

Forming the inner product of (3.1) and (3.2) with $G^*_{\tilde{u}}$ and G_{u^0}, respectively, yields

$$\langle[L(u^0)+\delta\gamma(u^0)]G_{u^0}, G^*_{\tilde{u}}\rangle=G^*_{\tilde{u}} \tag{3.3}$$

and

$$\langle G_{u^0}, [L^*(\tilde{u})+\delta^*\gamma^*(\tilde{u})]G^*_{\tilde{u}}\rangle=G_{u^0}. \tag{3.4}$$

Using the duality relationship (2.7) and subtracting (3.4) from (3.3) leads to

$$G^*_{\tilde{u}}=G_{u^0}+\langle G_{u^0}, [L^*(u^0)+\delta^*\gamma^*(u^0)-L^*(\tilde{u})-\delta^*\gamma^*(\tilde{u})]G^*_{\tilde{u}}\rangle. \tag{3.5}$$

Relationship (3.5) is a closed-form non-linear integrodifferential equation satisfied by the forward propagator $G^*_{\tilde{u}}$. A similar procedure can be applied to the systems

$$[L(\tilde{u})+\delta\gamma(\tilde{u})]G_{\tilde{u}}=\delta(t-t') \tag{3.6}$$

and

$$[L^*(u^0)+\delta^*\gamma^*(u^0)]G^*_{u^0}=\delta(t-t'') \tag{3.7}$$

to obtain the closed-form non-linear integrodifferential equation for the backward propagator $G_{\tilde{u}}$:

$$G_{\tilde{u}}=G^*_{u^0}+\langle[L(u^0)+\delta\gamma(u^0)-L(\tilde{u})-\delta\gamma(\tilde{u})]G_{\tilde{u}}, G^*_{u^0}\rangle. \tag{3.8}$$

The non-linear character of (3.5) and (3.8) stems from the fact that the operators $\{L^*(\tilde{u}), \gamma^*(\tilde{u})\}$ and $\{L(\tilde{u}), \gamma(\tilde{u})\}$ depend non-linearly on $\tilde{u}$, while $\tilde{u}$ depends, in turn, on the propagators $G^*_{\tilde{u}}$ or $G_{\tilde{u}}$ via (2.13) or (2.14), respectively. The integrodifferential character of (3.5) and (3.8) stems from the combination of the integral character of the inner product and the differential character of L^* and L, respectively. In several important particular problems, though, such as the Burgers and the Korteweg–de Vries equations, (3.5) and/or (3.8) become purely integral equations (Cacuci and Karakashian 1988). Note also that (3.5) and (3.8) are *exact* and their non-linear character stems not from closure approximations but reflects *exactly* the non-linearities of the original system (2.1).

Because they retain the full non-linear information contained in the original problem (2.1), equations (3.5) and (3.8) may be difficult to solve in practice. Moreover, they yield directly only the propagator, so the solution to the original system must subsequently be computed from the convolution expressions given by (2.13) or (2.14).

An alternative, and possibly more efficient, approach is to obtain an integrodifferential equation, similar to (3.5) or (3.8), for the solution $\tilde{u}$ itself. For this purpose, we note that since the function u^0 is known, equation (3.7) can, in principle, be solved to obtain $G^*_{u^0}$ as the inverse of the linear operator $L^*(u^0)+\delta^*\gamma^*(u^0)$. Then, the solution

$\tilde{u}$ can be obtained by using (3.7), (2.6) and (2.1), the linearity of the operators L^*, and γ^*, and performing the following sequence of operations:

$$\begin{aligned}\tilde{u} &= \langle \tilde{u}, [L^*(u^0) + \delta^* \gamma^*(u^0)] G^*_{u^0} \rangle \\ &= \langle \tilde{u}, [L^*(\tilde{u}) + \delta^* \gamma^*(\tilde{u})] G^*_{u^0} \rangle + \langle \tilde{u}, [L^*(u^0) + \delta^* \gamma^*(u^0) - L^*(\tilde{u}) - \delta^* \gamma^*(\tilde{u})] G^*_{u^0} \rangle \\ &= \langle [L(\tilde{u}) + \delta\gamma(\tilde{u})]\tilde{u}, G^*_{u^0} \rangle + \langle \tilde{u}, [L^*(u^0) + \delta^* \gamma^*(u^0) - L^*(\tilde{u}) - \delta\gamma^*(\tilde{u})] G^*_{u^0} \rangle \\ &= \langle f + \delta g, G^*_{u^0} \rangle + \langle \tilde{u}, [L^*(u^0) + \delta^* \gamma^*(u^0) - L^*(\tilde{u}) - \delta^* \gamma^*(\tilde{u})] G^*_{u^0} \rangle . \qquad (3.9)\end{aligned}$$

The last equality in (3.9) represents, in general, an integrodifferential equation for the solution $\tilde{u}$ in terms of the known sources f and g, and the known propagator $G^*_{u^0}$. However, as discussed in the foregoing, equation (3.9) may reduce to a purely integral equation; two such instances of particular importance are the Burgers and the Korteweg–de Vries equations. In these cases, very efficient and accurate methods of numerical and functional analysis can be used to solve, as shown by Cacuci and Karakashian (1988), these otherwise difficult initial/boundary value problems.

4. Conclusions

In this work, we have extended to multicomponent (i.e. vector) problems the canonical formalism originally introduced by Cacuci *et al* (1988) for solving non-linear problems in terms of propagators that generalise the customary Green functions in linear theory. Fundamental to the development of our formalism as the non-linear analogue of the Green function method are the operators $L(u)$ and $L^*(u)$; these operators are obtained by functionally integrating the Gâteaux derivative $N'(u)$, of the original non-linear operator $N(u)$, and its adjoint $[N'(u)]^*$, respectively. Also essential in developing our formalism is the observation that the relationship $[L(u)+\delta\gamma(u)]u = N(u)+\gamma\Gamma(u)$ is satisfied by $L(u)$, $\gamma(u)$, but not by the variational operators $N'(u)$ and $\gamma'(u)$. Thus, the forward and backward propagators G^*_u and G_u, which are solutions of equations involving the operators $L^*(u)$ and $L(u)$, carry all the information needed to solve the original non-linear equation. This is the very reason that the propagators G^*_u and G_u generalise the customary Green functions from linear theory; in particular, when the original problem is linear, G^*_u and G_u reduce to the usual Green functions, since, in this case, the operators L and L^* become independent of u and, consequently, so do G^*_u and G_u.

We have shown that the advanced and retarded propagators G^*_u and G_u satisfy a reciprocity relationship analogous to that satisfied by the customary Green functions. We have further shown that these propagators can be obtained as solutions of equations that are, in general, both non-linear and integrodifferential in character; note, though, that the order of the highest derivative appearing in these equations will always be lower than the order of the highest derivative appearing in the original non-linear system. Furthermore, in many important particular cases, such as the Burgers and the Korteweg–de Vries equations, the integrodifferential equation for one or both of the propagators reduces to a purely integral equation.

We further noted that the integrodifferential (or integral) equations for the propagators are *exact*, and their non-linear character reflects *exactly* the non-linearities present in the original system. This is in contradistinction to non-linearities that may appear in the expressions of the Green functions in many-body and field theories, where such non-linearities are not intrinsic to these theories but are introduced as a result of approximations needed to close the respective equations.

Using the advanced or retarded propagators, we have converted the solution of the original boundary/initial value problem into an integral form. In principle, such a conversion is always advantageous, even if the resulting integral form is non-linear. This is both because (*a*) the contraction principle and/or fixed-points theorems could be applied to this integral form (but not to the original non-linear boundary/initial value problem) to prove existence and uniqueness, and (*b*) most numerical analysis and computational methods are comparatively more mature and less difficult to implement for integral equations than for differential ones. Note, though, that because the non-linearities of the original problem are inherently and exactly incorporated into the integral equations produced by our formalism, these integral equations may still be very difficult to solve in practice.

The *formal* character of the derivations underlying our formalism is underscored by the fact that we have not addressed, in general, the issues of existence, well-posedness, continuous dependence on data, etc; these issues can only be addressed in detail for each specific problem and functional setting.

Appendix 1

A simple illustration that nevertheless highlights the main features of applying the general formalism presented in §§ 2 and 3 can be performed by considering the scalar Riccati equation

$$N(u) = \mathrm{d}u/\mathrm{d}t + u^2 = 0 \tag{A1.1}$$

with initial condition

$$\lim_{t \downarrow 0} u(t) = u_0 > 0. \tag{A1.2}$$

Solving this Riccati equation in $L_2[0, t_f]$ by standard procedures gives the unique solution

$$u(t) = \frac{u_0}{u_0 t + 1}. \tag{A1.3}$$

We remark that here, as in the following two appendices, we use the L_2 setting only to maintain a consistently unified framework with the theoretic §§ 2 and 3. In the remainder of this appendix, we will apply the general formalism developed in §§ 2 and 3 to solve this Riccati equation and recover the solution (A1.3).

Comparing (A1.1) and (A1.2) with (2.1) leads to the identifications: $\Omega \to [0, t_f]$, $\delta\Gamma(u) \to \delta(t)u(t), f \to 0$, and $\delta g \to \delta(t)u_0$, respectively. Besides (A1.1), we also consider the following equations:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{u_0}{u_0 t + 1} = 0 \tag{A1.4}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{u_0}{u_0 t + 1} - u + \frac{u_0}{u_0 t + 1} = 0 \tag{A1.5}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{u_0^2}{(u_0 t + 1)^2} = 0 \tag{A1.6}$$

for $t \in (0, t_f)$, all subject to the initial condition (A1.2). Note that each of these equations admits a unique solution in $L_2[0, t_f]$ and this solution is, in all cases, given by (A1.3). Of course, equations (A1.4), (A1.5) and (A1.6) are all equivalent to (A1.1). The purpose for considering these four equivalent forms of the same equation is to illustrate that although the application of the general formalism presented in §§ 2 and 3 will yield distinct expressions for the respective operators $L(u)$ and hence for the respective propagators $G(t, t')$, these propagators will all yield the correct, unique solution (A1.3) in each case.

Performing the operations leading to (2.8) on (A1.1), (A1.4), (A1.5), and (A1.6) shows that the backward propagators $G(t, t')$ for these four equations are the solutions of

$$\frac{dG}{dt} + uG = \delta(t - t') \tag{A1.7}$$

$$\frac{dG}{dt} + \frac{u_0}{u_0 t + 1} G = \delta(t - t') \tag{A1.8}$$

$$\frac{dG}{dt} + \frac{u_0}{u_0 t + 1} G - G = \delta(t - t') \tag{A1.9}$$

$$\frac{dG}{dt} = \delta(t - t') \tag{A1.10}$$

respectively; all of these propagators are subject to the initial condition

$$G(0, t') = 0. \tag{A1.11}$$

The solutions to (A1.7)-(A1.11) are

$$G(t, t') = H(t - t') \exp\left(- \int_{t'}^{t} u(s)\, ds \right) \tag{A1.12}$$

$$G(t, t') = H(t - t') \exp\left(- \int_{t'}^{t} \frac{u_0}{s u_0 + 1}\, ds \right) = H(t - t') \frac{u_0 t' + 1}{u_0 t + 1} \tag{A1.13}$$

$$G(t, t') = H(t - t') \exp[-(t' - t)] \frac{u_0 t' + 1}{u_0 t + 1} \tag{A1.14}$$

$$G(t, t') = H(t - t') \tag{A1.15}$$

respectively, where $H(t - t')$ is the Heaviside function

$$H(t - t') = \begin{cases} 1 & \text{for } t - t' \geq 0 \\ 0 & \text{for } t - t' < 0. \end{cases} \tag{A1.16}$$

Therefore, equation (2.14) yields

$$u(t) = \int_0^t G(t, t')(f(t') + \delta(t') u_0)\, dt' \tag{A1.17}$$

as the general, although implicit, expression for the solution of each of (A1.1), (A1.4), (A1.5), and (A1.6). Replacing equations (A1.12)-(A1.15) in (A1.17), and noting that the external source $f(t')$ is non-zero only for (A1.5), leads to

$$u(t)=\exp\left(-\int_0^t u(s)\,\mathrm{d}s\right) \tag{A1.18}$$

$$u(t)=\frac{u_0}{u_0t+1} \tag{A1.19}$$

$$\begin{aligned} u(t)&=-\int_0^t G(t,t')\frac{u_0}{u_0t'+1}\,\mathrm{d}t'+e^t\frac{u_0}{u_0t+1} \\ &=-\int_0^t \exp[-(t'-t)]\frac{u_0}{u_0t+1}\,\mathrm{d}t'+\exp(t)\frac{u_0}{u_0t+1}=\frac{u_0}{u_0t+1} \end{aligned} \tag{A1.20}$$

$$u(t)=-\int_0^t H(t-t')\frac{u_0^2}{(u_0t'+1)^2}\,\mathrm{d}t'+G_{u_0}(0,t)=\frac{u_0}{u_0t+1} \tag{A1.21}$$

respectively, as solutions to equations (A1.1), (A1.4), (A1.5) and (A1.6). Note that (A1.19)-(A1.21) are identical to the explicit solution (A1.3), while (A1.18) solves the original equation implicitly.

Appendix 2

In this appendix, we illustrate the application of the general formalism presented in §§ 2 and 3 to a system of two non-linear equations. This example is taken from the kinetic theory of gases and is known as the homogeneous Carleman system (Carleman 1957) for two gases of concentration u_1 and u_2:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t}=-u_1^2+u_2^2 \qquad u_1(0)=u_{1,0}>0 \tag{A2.1}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t}=-u_2^2+u_1^2 \qquad u_2(0)=u_{2,0}>0. \tag{A2.2}$$

In the real space $L_2[0,t_f]\times L_2[0,t_f]$, the solution of this Carleman system is readily obtained by introducing the transformation

$$u_1+u_2=n \qquad u_1-u_2=j \tag{A2.3}$$

in (A2.1) and (A2.2); the resulting equations are

$$\mathrm{d}n/\mathrm{d}t=0 \tag{A2.4}$$

and

$$\mathrm{d}j/\mathrm{d}t=-2nj \tag{A2.5}$$

with solutions

$$n=u_{1,0}+u_{2,0}=C \tag{A2.6}$$

and

$$j=(u_{1,0}-u_{2,0})\,\mathrm{e}^{-2Ct}. \tag{A2.7}$$

Using equations (A2.6) and (A2.7) in (A2.3) gives the solutions $u_1(t)$ and $u_2(t)$ of the Carleman system as

$$u_1(t)=\tfrac{1}{2}[C+(u_{1,0}-u_{2,0})\,\mathrm{e}^{-2Ct}] \tag{A2.8}$$

and

$$u_2(t)=\tfrac{1}{2}[C-(u_{1,0}-u_{2,0})\,\mathrm{e}^{-2Ct}]. \tag{A2.9}$$

Introducing the two-component vectors $u=(u_1, u_2)$, $N=(N_1, N_2)$, $\Gamma=(\Gamma_1, \Gamma_2)$, $f=(0,0)$, and $g=(u_{1,0}, u_{2,0})$, we recast (A2.1) and (A2.2) in the form of (2.1), i.e.

$$\begin{aligned} N_1(u)+\delta\Gamma_1(u)&=\frac{\mathrm{d}u_1}{\mathrm{d}t}+u_1^2-u_2^2+\delta(t)u_1=\delta(t)u_{1,0}\\ N_2(u)+\delta\Gamma_2(u)&=\frac{\mathrm{d}u_2}{\mathrm{d}t}+u_2^2-u_1^2+\delta(t)u_2=\delta(t)u_{2,0}. \end{aligned} \tag{A2.10}$$

The application of the general formalism developed in § 2 follows now in a straightforward manner: we have

$$\begin{aligned} N'(u)+\delta\Gamma'(u)&\\ &=\begin{pmatrix} \mathrm{d}N_1(u)/\mathrm{d}u_1 & \mathrm{d}N_1(u)/\mathrm{d}u_2\\ \mathrm{d}N_2(u)/\mathrm{d}u_1 & \mathrm{d}N_2(u)/\mathrm{d}u_2\end{pmatrix}+\delta(t)\begin{pmatrix} \mathrm{d}\Gamma_1(u)/\mathrm{d}u_1 & \mathrm{d}\Gamma_1(u)/\mathrm{d}u_2\\ \mathrm{d}\Gamma_2(u)/\mathrm{d}u_1 & \mathrm{d}\Gamma_2(u)/\mathrm{d}u_2\end{pmatrix}\\ &=\begin{pmatrix} \mathrm{d}/\mathrm{d}t+2u_1 & -2u_2\\ -2u_1 & \mathrm{d}/\mathrm{d}t+2u_2\end{pmatrix}+\delta(t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \end{aligned} \tag{A2.11}$$

and, replacing u by εu in (A2.11) and integrating over ε from 0 to 1, we obtain

$$L(u)+\delta\gamma(u)=\begin{pmatrix} \mathrm{d}/\mathrm{d}t+u_1 & -u_2\\ -u_1 & \mathrm{d}/\mathrm{d}t+u_2\end{pmatrix}+\delta(t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{A2.12}$$

The formal adjoint, $N'^*(u)$, of $N'(u)$ is determined by the usual linear duality so that

$$N'^*(u)+\delta^*\gamma^*(u)=\begin{pmatrix} -\mathrm{d}/\mathrm{d}t+2\bar{u}_1 & -2\bar{u}\\ -2\bar{u}_2 & -\mathrm{d}/\mathrm{d}t+2\bar{u}_2\end{pmatrix}+\delta(t-t_f)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{A2.13}$$

where the bar denotes complex conjugation. Since we work in real spaces, $u_1=\bar{u}_1$ and $u_2=\bar{u}_2$. Replacing u by εu in (A2.13) and integrating over ε from 0 to 1 gives

$$L^*(u)+\delta^*\gamma^*(u)=\begin{pmatrix} -\mathrm{d}/\mathrm{d}t+u_1 & -u_1\\ -u_2 & -\mathrm{d}/\mathrm{d}t+u_2\end{pmatrix}+\delta(t-t_f)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{A2.14}$$

The Carleman system can now be solved by using either the forward or the backward propagators. In the following, we will use the backward propagator $G(t, t')$. This propagator satisfies (2.8), where $L(u)+\delta\gamma(u)$ is given by (A2.12):

$$\begin{pmatrix} \mathrm{d}/\mathrm{d}t+u_1 & -u_2\\ -u_1 & \mathrm{d}/\mathrm{d}t+u_2\end{pmatrix}\begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\delta(t-t') \tag{A2.15}$$

subject to the initial conditions

$$G_{ij}(0, t')=0 \qquad \text{for } i,j=1,2. \tag{A2.16}$$

In component form, (A2.15) is

$$\mathrm{d}G_{11}/\mathrm{d}t+u_1G_{11}-u_2G_{21}=\delta(t-t') \tag{A2.17}$$

$$\mathrm{d}G_{12}/\mathrm{d}t+u_1G_{12}-u_2G_{22}=0 \tag{A2.18}$$

$$\mathrm{d}G_{21}/\mathrm{d}t-u_1G_{11}+u_2G_{21}=0 \tag{A2.19}$$

$$\mathrm{d}G_{22}/\mathrm{d}t-u_1G_{12}+u_2G_{22}=\delta(t-t'). \tag{A2.20}$$

Adding equations (A2.17) and (A2.19) and integrating over t gives

$$G_{11}+G_{21}=H(t-t') \tag{A2.21}$$

the integration constant being zero in view of (A2.16). Similarly, adding equations (A2.18) and (A2.20) and integrating gives

$$G_{22}+G_{12}=H(t-t'). \tag{A2.22}$$

Using equations (A2.21) and (A2.22) in (A2.17) and (A2.19), respectively, and performing the respective integrations leads to

$$G_{11}(t,t')=H(t-t')\left(\mathrm{e}^{C(t'-t)}+\int_{t'}^{t}\mathrm{e}^{C(\tau-t)}u_2(\tau)\,\mathrm{d}\tau\right) \tag{A2.23}$$

$$G_{21}(t,t')=H(t-t')\left(1-\mathrm{e}^{C(t'-t)}-\int_{t'}^{t}\mathrm{e}^{C(\tau-t)}u_2(\tau)\,\mathrm{d}\tau\right) \tag{A2.24}$$

$$G_{22}(t,t')=H(t-t')\left(\mathrm{e}^{C(t'-t)}+\int_{t'}^{t}\mathrm{e}^{C(\tau-t)}u_1(\tau)\,\mathrm{d}\tau\right) \tag{A2.25}$$

$$G_{12}(t,t')=H(t-t')\left(1-\mathrm{e}^{C(t'-t)}-\int_{t'}^{t}\mathrm{e}^{C(\tau-t)}u_1(\tau)\,\mathrm{d}\tau\right). \tag{A2.26}$$

Using equations (A2.23)-(A2.26) in (A2.14) gives the solution $u=(u_1, u_2)$ to the Carleman system as

$$\begin{aligned}
u_1(t)&=u_{1,0}G_{11}(t,0)+u_{2,0}G_{12}(t,0)\\
&=u_{1,0}\left(\mathrm{e}^{-Ct}+\int_0^t\mathrm{e}^{C(\tau-t)}u_2(\tau)\,\mathrm{d}\tau\right)\\
&\quad+u_{2,0}\left(1-\mathrm{e}^{-Ct}-\int_0^t\mathrm{e}^{C(\tau-t)}u_1(\tau)\,\mathrm{d}\tau\right)\\
&=u_{1,0}\left(\mathrm{e}^{-Ct}+\tfrac{1}{2}-\frac{u_{1,0}}{C}\mathrm{e}^{-Ct}+\frac{u_{1,0}-u_{2,0}}{2C}\mathrm{e}^{-2Ct}\right)\\
&\quad+u_{2,0}\left(-\mathrm{e}^{-Ct}-\tfrac{1}{2}+\frac{u_{2,0}}{C}\mathrm{e}^{-Ct}+\frac{u_{1,0}-u_{2,0}}{2C}\mathrm{e}^{-2Ct}\right)\\
&=\tfrac{1}{2}C+\tfrac{1}{2}(u_{1,0}-u_{2,0})\,\mathrm{e}^{-2Ct}
\end{aligned} \tag{A2.27}$$

and

$$\begin{aligned}
u_2(t)&=u_{1,0}G_{21}(t,0)+u_{2,0}G_{22}(t,0)\\
&=u_{1,0}\left(1-\mathrm{e}^{-Ct}-\int_0^t\mathrm{e}^{C(\tau-t)}u_2(\tau)\,\mathrm{d}\tau\right)\\
&\quad+u_{2,0}\left(\mathrm{e}^{-Ct}+\int_0^t\mathrm{e}^{C(\tau-t)}u_1(\tau)\,\mathrm{d}\tau\right)\\
&=u_{1,0}\left(-\mathrm{e}^{-Ct}+\frac{u_{1,0}}{C}\mathrm{e}^{-Ct}-\frac{u_{1,0}-u_{2,0}}{2C}\mathrm{e}^{-2Ct}\right)\\
&\quad+u_{2,0}\left(\mathrm{e}^{-Ct}+\tfrac{1}{2}-\frac{u_{2,0}}{C}\mathrm{e}^{-Ct}-\frac{u_{1,0}-u_{2,0}}{2C}\mathrm{e}^{-2Ct}\right)\\
&=\tfrac{1}{2}C-\tfrac{1}{2}(u_{1,0}-u_{2,0})\,\mathrm{e}^{-2Ct}.
\end{aligned} \tag{A2.28}$$

As expected, equations (A2.27) and (A2.28) are identical to (A2.8) and (A2.9), respectively.

Just as in appendix A, it is instructive to consider the following equivalent form of the Carleman system, obtained by using equation (2.6) in (A2.1) and (A2.2):

$$\begin{aligned} du_1/dt &= -C(u_1 - u_2) \\ du_2/dt &= -C(u_2 - u_1). \end{aligned} \tag{A2.29}$$

This equivalent representation of the Carleman system leads to

$$L(u) = \begin{pmatrix} d/dt + C & -C \\ -C & d/dt + C \end{pmatrix} \tag{A2.30}$$

$$L^*(u) = \begin{pmatrix} -d/dt + C & -C \\ -C & -d/dt + C \end{pmatrix} \tag{A2.31}$$

and, hence, to the following system for the backward propagator $G(t, t')$:

$$dG_{11}/dt + CG_{11} - G_{21}C = \delta(t - t') \tag{A2.32}$$

$$dG_{12}/dt + CG_{12} - CG_{22} = 0 \tag{A2.33}$$

$$dG_{21}/dt - CG_{11} + CG_{21} = 0 \tag{A2.34}$$

$$dG_{22}/dt - CG_{12} + CG_{22} = \delta(t - t'). \tag{A2.35}$$

The solution of equations (A2.32)-(A2.35) is

$$G_{11}(t, t') = \tfrac{1}{2}H(t - t')(1 + e^{2C(t'-t)}) \tag{A2.36}$$

$$G_{21}(t, t') = \tfrac{1}{2}H(t - t')(1 - e^{2C(t'-t)}) \tag{A2.37}$$

$$\begin{aligned} G_{22}(t, t') &= \tfrac{1}{2}H(t - t')(1 + e^{2C(t'-t)}) \\ &= G_{11}(t, t') \end{aligned} \tag{A2.38}$$

$$\begin{aligned} G_{12}(t, t') &= \tfrac{1}{2}H(t - t')(1 - e^{2C(t'-t)}) \\ &= G_{21}(t, t'). \end{aligned} \tag{A2.39}$$

Using now equations (A2.36)-(A2.39) in (2.14) leads to

$$\begin{aligned} u_1(t) &= u_{1,0}G_{11}(t, 0) + u_{2,0}G_{12}(t, 0) \\ &= u_{1,0}\,\tfrac{1}{2}(1 + e^{-2Ct}) + u_{2,0}\,\tfrac{1}{2}(1 - e^{-2Ct}) \\ &= \tfrac{1}{2}C + \tfrac{1}{2}(u_{1,0} - u_{2,0})\,e^{-2Ct} \end{aligned} \tag{A2.40}$$

and

$$\begin{aligned} u_2(t) &= u_{1,0}G_{21}(t, 0) + u_{2,0}G_{22}(t, 0) \\ &= u_{1,0}\,\tfrac{1}{2}(1 - e^{-2Ct}) + u_{2,0}\,\tfrac{1}{2}(1 + e^{-2Ct}) \\ &= \tfrac{1}{2}C - \tfrac{1}{2}(u_{1,0} - u_{2,0})\,e^{-2Ct}. \end{aligned} \tag{A2.41}$$

Thus, although the operators L and L^*, and consequently the propagator $G(t, t')$, for the system (A2.29) are distinct from their counterparts for equations (A2.1) and (A2.2), the final expression obtained for the solution u (i.e. equations (A2.40) and (A2.41) and equations (A2.27) and (A2.28), respectively) is the same in both cases. This outcome is similar to that obtained for the scalar case presented in appendix 1. Note also that the system (A2.29) is linear in u (in contradistinction to the system (A2.1) and (A2.2), which is non-linear) and the propagator $G(t, t')$, i.e. equations (A2.36)-(A2.39), given by the general formalism of § 2 is, in this case, the same as the Green function that would have been obtained by using the well known methods of linear theory.

Appendix 3

In this appendix, we apply the canonical formalism presented in §§ 2 and 3 to the matrix Riccati equation

$$\mathrm{d}u/\mathrm{d}t = b + Bu - uc^{\mathrm{T}}u \tag{A3.1}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad c^{\mathrm{T}} = (c_1, c_2). \tag{A3.2}$$

Considered as an evolution equation in $L_2[0, t_f] \times L_2[0, t_f]$ subject to the initial condition

$$\lim_{t\downarrow 0} u(t) = \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} \qquad u_{10} > 0,\ u_{20} > 0 \tag{A3.3}$$

equation (A3.1) admits the unique solution

$$u_1 = \frac{1}{\sqrt{\beta}} \frac{c_2(b_2u_{10} - b_1u_{20}) + b_1[v_0 \cosh(t\sqrt{\beta}) + \sqrt{b}\sinh(t\sqrt{\beta})]}{\sqrt{\beta}\cosh(t\sqrt{\beta}) + v_0\sinh(t\sqrt{\beta})} \tag{A3.4}$$

$$u_2 = \frac{1}{\sqrt{\beta}} \frac{c_1(b_1u_{20} - b_2u_{10}) + b_2[v_0 \cosh(t\sqrt{\beta}) + \sqrt{b}\sinh(t\sqrt{\beta})]}{\sqrt{\beta}\cosh(t\sqrt{\beta}) + v_0\sinh(t\sqrt{\beta})} \tag{A3.5}$$

where

$$v_0 \equiv c_1u_{10} + c_2u_{20} \tag{A3.6}$$

and

$$\beta \equiv c_1b_1 + c_2b_2. \tag{A3.7}$$

For this Riccati equation, the equations for the forward and backward propagators are

$$L^*(u)G_u^*(t;\, t')$$
$$= \begin{pmatrix} -\mathrm{d}/\mathrm{d}t + c_1u_1 + \frac{1}{2}c_2u_2 & \frac{1}{2}c_1u_2 \\ \frac{1}{2}c_2u_1 & -\mathrm{d}/\mathrm{d}t + \frac{1}{2}c_1u_1 + c_2u_2 \end{pmatrix} \begin{pmatrix} G_{11}^* & G_{12}^* \\ G_{21}^* & G_{22}^* \end{pmatrix}$$
$$= \delta(t - t_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A3.8}$$

$$G_u^*(t;\, t') = 0 \qquad \text{at } t = t_f \qquad \text{i.e. for } t > t' \tag{A3.9}$$

and

$$L(u)G_u(t,\, t'')$$
$$= \begin{pmatrix} \mathrm{d}/\mathrm{d}t + c_1u_1 + \frac{1}{2}c_2u_2 & \frac{1}{2}c_2u_1 \\ \frac{1}{2}c_1u_2 & \mathrm{d}/\mathrm{d}t + \frac{1}{2}c_1u_1 + c_2u_2 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$
$$= \delta(t - t'')\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A3.10}$$

$$G_u(t,\, t'') = 0 \qquad \text{at } t = 0 \qquad \text{i.e. for } t < t'' \tag{A3.11}$$

respectively.

Note that equations (A3.8) and (A3.10) are linear in G_u^* and G_u, respectively, so they can be solved by the standard methods for first-order matrix differential equations. Thus, the fundamental matrix for (A3.10) is given by

$$\Phi(t,\tau) \equiv I - \int_\tau^t A(\sigma_1)\,d\sigma_1 + \int_\tau^t A(\sigma_1)\,d\sigma_1 \int_\tau^{\sigma_1} A(\sigma_2)\,d\sigma_2 - \ldots \tag{A3.12}$$

where I is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A3.13}$$

and

$$A \equiv \begin{pmatrix} c_1u_1 + \frac{1}{2}c_2u_2 & \frac{1}{2}c_2u_1 \\ \frac{1}{2}c_1u_2 & \frac{1}{2}c_1u_1 + c_2u_2 \end{pmatrix}. \tag{A3.14}$$

Hence, the solution to equations (A3.10) and (A3.11) is

$$G_u(t, t'') = H(t - t'')\Phi(t, t''). \tag{A3.15}$$

A similar procedure gives the solution of (A3.8) and (A3.9) as

$$G_u^*(t, t') = H(t' - t)\Phi^{\mathrm{T}}(t', t) \tag{A3.16}$$

where the symbol T denotes transposition. Equations (A3.15) and (A3.16) directly verify that the forward and backward propagators G_u^* and G_u for the Riccati equation satisfy the reciprocity relation (2.12).

The solution u of the Riccati equation is obtained in terms of the forward propagator G_u^* by following the operations indicated in (2.13). This gives

$$\begin{aligned} u(t) &= [G_u^*(0,t)]^{\mathrm{T}}\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} + \int_0^{t_f} [G_u^*(\tau,t)]^{\mathrm{T}}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} d\tau \\ &= [\Phi(t,0)]\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} + \int_0^t [\Phi(t,\tau)]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} d\tau. \end{aligned} \tag{A3.17}$$

In terms of the backward propagator G_u, the solution u becomes

$$\begin{aligned} u(t) &= \int_0^{t_f} [G_u(t,\tau)]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} d\tau + [G_u(t,0)]\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} \\ &= \int_0^t [\Phi(t,\tau)]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} d\tau + [\Phi(t,0)]\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix}. \end{aligned} \tag{A3.18}$$

The result is identical with that obtained from (A3.17), as expected.

The validity of (A3.17) as the solution to the Riccati equation can be readily verified by direct substitution in (A3.1):

$$\begin{aligned} \frac{du}{dt} &= [\Phi(t,t)]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \int_0^t \left[\frac{d\Phi}{dt}\right]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} d\tau + \left[\frac{d\Phi(t,0)}{dt}\right]\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} \\ &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - A(t)\left\{\left[\int_0^t [\Phi(t,\tau)]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} d\tau + [\Phi(t,0)]\begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix}\right]\right\} \\ &= b - A(t)u(t) \\ &= b - uc^{\mathrm{T}}u \qquad \text{QED.} \end{aligned}$$

Under certain conditions, the Peano-Baker series representation of the fundamental matrix $\Phi(t, \tau)$ given in (A3.12) can be summed in a closed form. Such is the case, for example, if $c_1 = 0$, i.e., $c^T = (0, c_2)$; then, the propagators can also be expressed compactly in terms of exponentials involving the functions u_1 and u_2. The backward propagator $G_u(t, t'')$, for example, becomes in this case

$$G(t, t'') = H(t - t'')\begin{pmatrix} G_{11}(t, t'') & G_{12}(t, t'') \\ G_{21}(t, t'') & G_{22}(t, t'') \end{pmatrix} \tag{A3.19}$$

where

$$G_{11}(t, t'') = \exp\left(\frac{c_2}{2}\int_t^{t''} u_2(\tau)\,\mathrm{d}\tau\right) \tag{A3.20}$$

$$G_{12}(t, t'') = \left[\int_t^{t''} \frac{c_2}{2} u_1(\tau) \exp\left(\frac{c_2}{2}\int_\tau^{t''} u_2(s)\,\mathrm{d}s\right)\mathrm{d}\tau\right] \exp\left(\frac{c_2}{2}\int_0^{t''} u_2(\tau)\,\mathrm{d}\tau\right) \tag{A3.21}$$

$$G_{21}(t, t'') = 0 \tag{A3.22}$$

$$G_{22}(t, t'') = \exp\left(c_2\int_t^{t''} u_2(\tau)\,\mathrm{d}\tau\right). \tag{A3.23}$$

Replacing (A3.19) in (A3.18) leads to a fixed-point integral equation for u that can be solved to recover—after tedious algebra—the explicit solution given by (A3.4) and (A3.5) when $c_1 = 0$.

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